

On Gauss-Kuzmin Statistics and the Transfer Operator for a Multidimensional Continued Fraction Algorithm: the Triangle Map

Thomas Garrity
 Department of Mathematics
 Williams College
 Williamstown, MA 01267
 email:tgarrity@williams.edu

Abstract

The Gauss-Kuzmin statistics for the triangle map (a type of multidimensional continued fraction algorithm) are derived by examining the leading eigenfunction of the triangle map's transfer operator. The technical difficulty is finding the appropriate Banach space of functions. We also show that, by thinking of the triangle map's transfer operator as acting on a one-dimensional family of Hilbert spaces, the transfer can be thought of as a family of nuclear operators of trace class zero.

1 Introduction

It is classical to think of continued fractions as iterations of the Gauss map on the unit interval and then to consider the underlying dynamical system. For example, the Gauss-Kuzmin statistics for continued fractions can be reduced to the study of the spectrum of the associated transfer operator

[15, 18, 22, 30, 34]. Most multidimensional continued fraction algorithms are iterations of a triangle \triangle (though the Jacobi-Perron algorithm acts on a square). For background on many types of multidimensional continued fraction algorithms, see Schweiger's *Multidimensional Continued Fractions* [35] and Karpenkov's *Geometry of Continued Fractions* [21]. Thus it is natural and standard to think of these as dynamical systems. This paper concentrates on the transfer operator for the triangle map [9, 13, 21, 29, 31] and then uses results on the spectrum of this operator to look at the triangle map's Gauss-Kuzmin statistics.

We will show that the transfer operator for the triangle map is

$$\mathcal{L}(f)(x, y) = \sum_{n=0}^{\infty} \frac{1}{(1 + nx + y)^3} f\left(\frac{1}{1 + kx + y}, \frac{x}{1 + kx + y}\right)$$

Note that though this transfer operator looks similar to the transfer operator for the Gauss map:

$$\sum \frac{1}{(n + x)^2} f\left(\frac{1}{n + x}\right),$$

there is one significant difference, namely that in the triangle transfer operator there is the term nx in the denominator, while the continued fraction transfer operator has the term $n + x$ in the denominator. This difference is what prevents us from applying the standard bounds for the continued fraction case to the triangle case. It is this difference that to a large extent makes this paper not simply an easy generalization of earlier work.

In this paper we find an appropriate Banach space of functions for which we can show that 1 is the largest eigenvalue, with one-dimensional eigenspace,

of the transfer operator. This can then be used to understand the Gauss-Kuzmin statistics for the triangle map. We then show, in analog to the Gauss map, that there is also a Hilbert space approach. We will see that in an appropriate sense the triangle map's transfer operator can be thought of as a nuclear operator. Unlike the continued fraction case, we cannot be working in a Hilbert space of square-integrable functions on the two-dimensional domain \triangle , as the natural eigenfunction with eigenvalue one is not square-integrable. Instead we will be thinking of the transfer operator as acting on a space of functions $f(x, y)$ that are only square-integrable with respect to the variable y . Thus we will be acting on a family of Hilbert spaces parameterized by the variable x .

There are two overall goals for this paper. First is to begin the spectral analysis for the transfer operator of the triangle map. Second, though, is to set-up the machinery for joint work with Ilya Amburg [3]. While the triangle map, to some extent, is just one example among many of possible multidimensional continued fraction algorithms, in [10, 11] it is shown that, by varying the triangle map in a natural way, a whole collection of both new and old multidimensional continued fraction algorithms can be generated, called triangle partition maps. Further, in that work, it is shown that these triangle partition maps generate a family of multidimensional continued fraction algorithms, called combination triangle partition maps, that in turn are shown to include many, if not most, known multidimensional continued fractions. In [3], it will be shown that the transfer operators for half

of the triangle partition maps have behavior similar to that of the triangle map. This will allow us to apply the results of this paper to this other work. More interestingly, we will see that the transfer operators of the other half of the triangle partition maps have remarkably different properties.

We would like to thank Ilya Amburg for a lot of help on this paper, including collaboration on the actual formulas for the Gauss-Kuzmin statistics. We would also like to thank L. Pedersen for many useful comments.

2 Transfer Operator for Triangle Map

Subdivide $\triangle = \{(x, y) \in \mathbb{R}^2 : 1 > x > y > 0\}$ into subtriangles

$$\triangle_k = \{(x, y) \in \triangle : 1 - x - ky \geq 0 > 1 - x - (k+1)y\}$$

Definition 1. *The triangle map $T : \triangle \rightarrow \triangle$ is defined by setting $T = T_k$ for $(x, y) \in \triangle_k$, and in turn setting*

$$T_k(x, y) = \left(\frac{y}{x}, \frac{1 - x - ky}{x} \right).$$

The point $(x, y) \in \triangle$ has triangle sequence (a_0, a_1, a_2, \dots) if

$$T^N(x, y) \in \triangle_{a_N}.$$

For more on the triangle map, see [9, 13, 21, 29, 31]

To define the triangle map's transfer operator, we first compute the Jacobian:

$$J(x, y) = \det \begin{pmatrix} \frac{\partial}{\partial x} \left(\frac{y}{x} \right) & \frac{\partial}{\partial y} \left(\frac{y}{x} \right) \\ \frac{\partial}{\partial x} \left(\frac{1-x-ky}{x} \right) & \frac{\partial}{\partial y} \left(\frac{1-x-ky}{x} \right) \end{pmatrix} = \det \begin{pmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ \frac{ky-1}{x^2} & -\frac{k}{x} \end{pmatrix} = \frac{1}{x^3},$$

for $(x, y) \in \triangle_k$.

Next we need to find the inverses $t_k : \triangle_k \rightarrow \triangle$ of the triangle map, one for each non-negative integer k . These are

$$t_k(x, y) = \left(\frac{1}{1 + kx + y}, \frac{x}{1 + kx + y} \right)$$

which can be checked by direct calculation,.

By definition, for any differentiable map T , the transfer operator is

$$\mathcal{L}_T(F)(p) = \sum_{q: T(q)=p} \frac{1}{\text{Jac}(T(q))} f(q).$$

This leads to

Proposition 2. *The transfer operator for the triangle map is*

$$\mathcal{L}(f)(x, y) = \sum_{n=0}^{\infty} \frac{1}{(1 + nx + y)^3} f\left(\frac{1}{1 + kx + y}, \frac{x}{1 + kx + y}\right).$$

This map is also be called the Perron-Frobenius operator or the Ruelle-Perron-Frobenius operator with respect to the Lebesgue measure $\text{d}x\text{d}y$, having the property that for all $f \in L^1(\text{d}x\text{d}y)$, and all measurable subsets $A \subset \triangle$, $\int_A \mathcal{L}_T(f) \text{d}x\text{d}y = \int_{T^{-1}A} f \text{d}x\text{d}y$. For more on transfer maps in general, see Baladi [5] and for continued fractions, see Iosifescu and Kraaikamp [18].

By direct calculation, for the function

$$f(x, y) = \frac{1}{x(1 + y)}$$

we have that

$$\mathcal{L}(f)(x, y) = f(x, y).$$

What we now must do is find appropriate vector spaces of functions on which the operator \mathcal{L} acts so that $f(x, y)$ is the leading eigenfunction, meaning that we would know enough about the spectrum of \mathcal{L} to be able to state that the largest eigenvalue is one, and that it has a one-dimensional eigenspace.

3 Banach Space Approach

We want to identify a Banach space of real-valued functions on Δ on which the transfer operator has leading eigenfunction $\frac{1}{x(1+y)}$ with eigenvalue one.

Let $\mathcal{C}(\Delta)$ be the vector space of real-valued continuous functions on the triangle Δ . Note that since Δ is open, the elements of $\mathcal{C}(\Delta)$ need not be bounded.

Set

$$V = \{f \in \mathcal{C}(\Delta) : \exists C \in \mathbb{R} \text{ such that } |xf(x, y)| < C, \forall (x, y) \in \Delta\}.$$

This is a Banach space under the norm

$$\|f\| = \sup_{(x, y) \in \Delta} |xf(x, y)|$$

Theorem 3. *The transfer operator is a continuous linear map from V to itself.*

Proof. Linearity is immediate. We will show for all $f \in V$ that

$$\|\mathcal{L}f(x, y)\| \leq 3\|f\|.$$

For any $f \in V$, let $C = \|f\|$, which means that for all $(x, y) \in \Delta$ we have

$$|xf(x, y)| < C.$$

Then

$$\begin{aligned}
|x\mathcal{L}f(x, y)| &= \left| \sum_{n=0}^{\infty} \left(\frac{x}{(1+nx+y)^3} \right) f\left(\frac{1}{1+nx+y}, \frac{x}{1+nx+y} \right) \right| \\
&\leq \sum_{n=0}^{\infty} \left| \left(\frac{x}{(1+nx+y)^3} \right) f\left(\frac{1}{1+nx+y}, \frac{x}{1+nx+y} \right) \right| \\
&= \sum_{n=0}^{\infty} \left| \left(\frac{x}{(1+nx+y)^2} \right) \left(\frac{1}{1+nx+y} \right) f\left(\frac{1}{1+nx+y}, \frac{x}{1+nx+y} \right) \right| \\
&\leq \sum_{n=0}^{\infty} \left(\frac{Cx}{(1+nx+y)^2} \right) \\
&= Cx \sum_{n=0}^{\infty} \left(\frac{1}{(1+nx+y)^2} \right) \\
&\leq Cx \left(\int_0^{\infty} \left(\frac{1}{((1+y)+xt)^2} \right) dt + \frac{1}{(1+y)^2} \right) \\
&\leq Cx \left(\frac{1}{x(1+y)} + 1 \right) \\
&\leq 3C,
\end{aligned}$$

giving us that the transform map is indeed a continuous map of the Banach space V to itself.

□

Before proving that the largest eigenvalue of \mathcal{L} is one, with multiplicity one, we need a few lemmas.

We have

Lemma 4. *For two functions $f, g \in V$, suppose $f(x, y) \leq g(x, y)$. Then for*

$$\mathcal{L}(f)(x, y) \leq \mathcal{L}(g)(x, y)$$

for all $(x, y) \in \Delta$.

Proof. This follows from looking at

$$0 \leq g(x, y) - f(x, y)$$

and then showing that

$$0 = \mathcal{L}(0) \leq \mathcal{L}(g - f)(x, y) = \mathcal{L}(g)(x, y) - \mathcal{L}(f)(x, y),$$

which is clear.

□

We will also need

Lemma 5. *For any $f \in V$, there is a positive constant B such that for all $(x, y) \in \Delta$, we have*

$$\frac{-B}{x(1+y)} \leq f(x, y) \leq \frac{B}{x(1+y)}.$$

Proof. Let $f \in V$ be any element in V . Then there is a B so that

$$|xf| < \frac{B}{2}.$$

(The reason for the $1/2$ will be clear in a moment.) Then we have

$$|f| < \frac{B}{2x} < \frac{B}{x(1+y)},$$

as desired.

□

Theorem 6. *The largest eigenvalue of $\mathcal{L} : V \rightarrow V$ is one, with multiplicity one.*

Proof. (The overall structure of this proof is similar to the corresponding result for the Gauss map. Again, what is new is finding the correct Banach space and using the fairly recent result of Messaoudi, Nogueira, and Schweiger [29] that the triangle map is ergodic.)

Let $f \in V$ be an eigenfunction of \mathcal{L} with eigenvalue λ .

We know there is a constant B such that for all $(x, y) \in \Delta$

$$\frac{-B}{x(1+y)} \leq f(x, y) \leq \frac{B}{x(1+y)}.$$

Then for all positive n ,

$$\frac{-B}{x(1+y)} \leq \mathcal{L}^{(n)} f(x, y) \leq \frac{B}{x(1+y)}$$

and hence

$$\frac{-B}{x(1+y)} \leq \lambda^n f(x, y) \leq \frac{B}{x(1+y)}.$$

Thus we need

$$|\lambda| \leq 1.$$

We now have to show that this eigenvalue has multiplicity one. To prove this from scratch would be hard, but using that the triangle map has been proven to be ergodic by Messaoudi, Nogueira, and Schweiger [29], the result follows immediately from Theorem 4.2.2 in Lasota and Mackey's *Probabilistic Properties of Deterministic Systems* [24] which states

Theorem. *Let (X, \mathcal{A}, μ) be a measure space, $S : X \rightarrow X$ a non-singular transformation, and P the Frobenius-Perron operator associated to S . If S is ergodic, then there is at most one stationary density f_* of P . Further, if*

there is a unique stationary density f_* of P and $f_*(x) > 0$, a.e., then S is ergodic.

Here P is our \mathcal{L} . This gives us our result.

□

It would be interesting to give an argument for this along the lines of the first few chapters in [23].

4 The Gauss-Kuzmin Distribution for Triangle Maps

(This section is joint work with Ilya Amburg, and is a special case of chapter 16 in [2]).

We would like to know the statistics behind a number's triangle sequence. Knowing now that the transfer operator has leading eigenvector $1/x(1+y)$, coupled with the work of Messoundi, Noguira and Schweiger [29] showing that the triangle map is ergodic, allows us to apply standard theorems to explicitly determine these statistics.

Let $(x, y) \in \triangle$ have triangle sequence (a_1, a_2, a_3, \dots) . We define

$$P_{n,k}(x, y) = \frac{\#\{a_i : a_i = k \text{ and } 1 \leq i \leq n\}}{n},$$

or, in other words, the percentage of the a_i that are equal to k in the first n terms of the triangle sequence. Provided the limit exists, we set

$$P_k(x, y) = \lim_{n \rightarrow \infty} p_{n,k}(x, y).$$

We know that $d\mu = \frac{12}{\pi^2 x(1+y)} dx dy$ is an invariant measure with respect to the triangle map $T : \Delta \rightarrow \Delta$. (The extra $12/\pi^2$ is just to make the measure of the domain Δ be one.) Then in direct analog to Theorem 9.14 in [21], we have

Theorem 7. *For almost all $(\alpha, \beta) \in \Delta$,*

$$P_k(\alpha, \beta) = \int_{\Delta_k} d\mu = \int_{\Delta_k} \frac{12}{\pi^2 x(1+y)} dx dy.$$

The proof is exactly analogous to that in [21].

Theorem 8. *For the triangle map T , we have that*

$$P(0) = 1 - \frac{6\text{Li}_2\left(\frac{1}{4}\right) + 12\log^2(2)}{\pi^2}$$

and, for $k > 0$,

$$\begin{aligned} P(k) = & \frac{6}{\pi^2} \left[\text{Li}_2\left(\frac{1}{(k_1)^2}\right) - \text{Li}_2\left(\frac{1}{(k+2)^2}\right) \right. \\ & \left. + 4\log^2(k+1) - 2\log^2\left(\frac{k+2}{k+1}\right) - 2\log(k(k+2))\log(k+1) \right] \end{aligned}$$

Here Li_2 is the dilogarithm function.

Proof. These are calculations, for which we used Mathematica. We have from the previous theorem that $P_k(\alpha, \beta)$ is independent of $(\alpha, \beta) \in \Delta$ for almost all elements in the domain. Then we have

$$\begin{aligned} P(0) &= \int_{\Delta_k} \frac{12}{\pi^2 x(1+y)} dx dy \\ &= \int_{\frac{1}{2}}^1 \left(\int_{1-x}^x \frac{12}{\pi^2 x(1+y)} dy \right) dx \\ &= 1 - \frac{6\text{Li}_2\left(\frac{1}{4}\right) + 12\log^2(2)}{\pi^2} \end{aligned}$$

and, for $k > 0$,

$$\begin{aligned}
P(k) &= \int_{\Delta_k} \frac{12}{\pi^2 x(1+y)} dx dy \\
&= \int_{\frac{1}{k+1}}^1 \left(\int_{\frac{1-x}{k+1}}^{\frac{1-x}{k}} \frac{12}{\pi^2 x(1+y)} dy \right) dx + \int_{\frac{1}{k+2}}^{\frac{1}{k+1}} \left(\int_{\frac{1-x}{k+1}}^x \frac{12}{\pi^2 x(1+y)} dy \right) dx \\
&= \frac{6}{\pi^2} \left[\text{Li}_2 \left(\frac{1}{(k+1)^2} \right) - \text{Li}_2 \left(\frac{1}{(k+2)^2} \right) \right. \\
&\quad \left. + 4 \log^2(k+1) - 2 \log^2 \left(\frac{k+2}{k+1} \right) - 2 \log(k(k+2)) \log(k+1) \right]
\end{aligned}$$

□

5 Attempts at nuclearity/ On various Hilbert spaces of functions

For the Gauss map, Mayer and Roepstroff [27] showed that the transfer operator is nuclear of trace class zero. This section shows that nontrivial analogs hold for the triangle map. The difficulty in part stems from that the function $1/x(1+y)$ is not in the Hilbert space $L^2(dx dy)$, in contrast to the leading eigenfunction $1/(1+x)$ of the Gauss map's transfer operator being in $L^2(dx)$. Instead, we will think of $1/x(1+y)$ as in $L^2(dy)$, treating the variable x as a parameter.

On the positive reals, set

$$dm(t) = \frac{t}{e^t - 1} dt.$$

Then set

$$\eta_k(s) = \frac{s^k e^{-s}}{(k+1)!}$$

$$\begin{aligned}
e_k(t) &= L_k^1(t) \\
E_k(x, y) &= \int_0^\infty \left(\frac{1}{x^2}\right) e^{-t(\frac{1-x+y}{x})} e_k(t) dm(t) \\
\mathcal{K}_T(\phi(x, t)) &= \int_0^\infty \frac{J_1(2\sqrt{st})}{\sqrt{st}} \frac{t}{e^t - 1} \phi(x, s) dm(s) \\
\widehat{\phi}(xy) &= \frac{1}{x} \int_{s=0}^\infty e^{-sy} \phi(x, s) dm(s) \\
\langle \alpha(s), \beta(s) \rangle &= \int_0^\infty \alpha(s) \beta(s) dm(s).
\end{aligned}$$

(Here $L_k^1(t)$ denotes the first Laguerre polynomial and J_1 denotes the Bessel function of order one.)

Our goal is to show that

$$\begin{aligned}
\mathcal{L}_T f(x, y) &= \mathcal{L}_T \widehat{\phi}(x, y) \\
&= \frac{1}{x^2} \int_0^\infty e^{-t(\frac{1-x+y}{x})} \mathcal{K}_T(\phi(x, t)) dm(t) \\
&= \sum_{k=0}^\infty \langle \phi(x, s), \eta_k(s) \rangle E_k(x, y)
\end{aligned}$$

After making a significant change of variables, we will see that our argument mirrors that given in [27].

We start with functions $f(x, y)$ with domain \triangle for which there is a $\phi(x, s)$ that is in $L^2(dm(s))$ such that $f = \widehat{\phi}$ and such that $\mathcal{L}_T(f)$ exists. Note that for $\phi(x, y) = \frac{1-e^{-y}}{y}$

$$\frac{1}{x(1+y)} = \widehat{\phi}(x, y);$$

thus such functions exist and include the function $1/x(1+y)$.

We have

$$\mathcal{L}_T(f)(x, y) = \sum_{n=0}^\infty \frac{1}{(1+nx+y)^3} f\left(\frac{1}{1+nx+y}, \frac{x}{1+nx+y}\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{(1+nx+y)^3} \widehat{\phi} \left(\frac{1}{1+nx+y}, \frac{x}{1+nx+y} \right).$$

Set

$$w = \frac{1+y}{x}.$$

It is this change of variables that will allow us to use [27].

Then we have

$$\begin{aligned} \frac{1}{(1+nx+y)^3} &= \frac{1}{x^3(n+w)^3} \\ \phi \left(\frac{1}{1+nx+y}, \frac{x}{1+nx+y} \right) &= \phi \left(\frac{1}{x(n+w)}, \frac{1}{n+w} \right) \end{aligned}$$

Then $\mathcal{L}_{T,\mu}(f)(x, y)$ is

$$\sum_{n=0}^{\infty} \frac{1}{x^3(n+w)^3} \widehat{\phi} \left(\frac{1}{x(n+w)}, \frac{1}{n+w} \right)$$

which is

$$\sum_{n=0}^{\infty} \frac{1}{x^3(n+w)^3} (x(n+w)) \int_0^{\infty} e^{\frac{-s}{n+w}} \phi(x, s) dm(s)$$

which is

$$\frac{1}{x^2} \int_0^{\infty} \sum_{n=0}^{\infty} \frac{1}{(n+w)^2} e^{\frac{-s}{n+w}} \phi(x, s) dm(s).$$

We want to rewrite

$$\sum_{n=0}^{\infty} \frac{1}{(n+w)^2} e^{\frac{-s}{n+w}}$$

to eliminate the dependence on n .

We have

$$\sum_{n=0}^{\infty} \frac{1}{(n+w)^2} e^{\frac{-s}{n+w}} = \sum_{n=0}^{\infty} \frac{1}{(n+w)^2} \sum_{k=0}^{\infty} \left(\frac{(-s)^k}{k!(n+w)^k} \right)$$

$$= \sum_{k=0}^{\infty} \left(\frac{(-s)^k}{k!} \right) \sum_{n=0}^{\infty} \frac{1}{(n+w)^{k+2}}$$

From standard arguments on the Lerch Zeta function, we know that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{(n+w)^{k+2}} &= \frac{1}{(k+1)!} \int_0^{\infty} \frac{t^{k+1}}{1-e^{-t}} e^{-wt} dt \\ &= \frac{1}{(k+1)!} \int_0^{\infty} t^k e^{-(w-1)t} dm(t). \end{aligned}$$

Thus

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{(n+w)^2} e^{\frac{-s}{n+w}} &= \sum_{k=0}^{\infty} \left(\frac{(-s)^k}{k!} \right) \frac{1}{(k+1)!} \int_0^{\infty} t^k e^{-(w-1)t} dm(t) \\ &= \int_0^{\infty} \sum_{k=0}^{\infty} \left(\frac{(-st)^k}{k!(k+1)!} \right) e^{-(w-1)t} dm(t) \\ &= \int_0^{\infty} \frac{J_1(2\sqrt{st})}{\sqrt{st}} e^{-(w-1)t} dm(t), \end{aligned}$$

where J_1 is the Bessel function of order one.

Then

$$\begin{aligned} \mathcal{L}_{T,\mu} f(x, y) &= \frac{1}{x^2} \int_0^{\infty} \int_0^{\infty} \frac{J_1(2\sqrt{st})}{\sqrt{st}} e^{-(w-1)t} \phi(x, s) dm(t) dm(s) \\ &= \frac{1}{x^2} \int_0^{\infty} \int_0^{\infty} \frac{J_1(2\sqrt{st})}{\sqrt{st}} e^{-t(\frac{1-x+y}{x})} \phi(x, s) dm(t) dm(s) \end{aligned}$$

This is why we set

$$\mathcal{K}_{T,\mu}(\phi(x, t)) = \int_0^{\infty} \frac{J_1(2\sqrt{st})}{\sqrt{st}} \phi(x, s) dm(s).$$

We have

$$\begin{aligned} \mathcal{L}_{T,\mu} f(x, y) &= \frac{1}{x^2} \int_0^{\infty} \int_0^{\infty} \frac{J_1(2\sqrt{st})}{\sqrt{st}} e^{-t(\frac{1-x+y}{x})} \phi(x, s) dm(s) dm(t) \\ &= \frac{1}{x^2} \int_0^{\infty} e^{-t(\frac{1-x+y}{x})} \mathcal{K}_{T,\mu}(\phi(x, t)) dm(t). \end{aligned}$$

We want an understanding of the operator $\mathcal{K}_{T,\mu}$.

From [27], we know that

$$\frac{J_1(2\sqrt{st})}{\sqrt{st}} = \sum_{k=0}^{\infty} L_k^1(t) \frac{s^k e^{-s}}{(k+1)!}$$

where $L_k^1(s)$ is a first Laguerre polynomial.

We set

$$e_k(s) = L_k^1(s), \eta_k(s) = \frac{s^k e^{-s}}{(k+1)!}$$

Both of these are actually functions of only one variable.

Then, also from [27], we have

$$\mathcal{K}_{T,\mu}(\phi(x, t)) = \sum_{k=0}^{\infty} \langle \phi(x, s), \eta_k(s) \rangle e_k(t)$$

as follows from

$$\begin{aligned} \mathcal{K}_{T,\mu}(\phi(x, t)) &= \int_0^{\infty} \frac{J_1(2\sqrt{st})}{\sqrt{st}} \phi(x, s) dm(s) \\ &= \int_0^{\infty} \sum_{k=0}^{\infty} L_k^1(t) \frac{s^k e^{-s}}{(k+1)!} \phi(x, s) dm(s) \\ &= \int_0^{\infty} \sum_{k=0}^{\infty} e_k(t) \eta_k(s) \phi(x, s) dm(s) \\ &= \sum_{k=0}^{\infty} e_k(t) \int_0^{\infty} \eta_k(s) \phi(x, s) dm(s) \\ &= \sum_{k=0}^{\infty} \langle \phi(x, s), \eta_k(s) \rangle e_k(t). \end{aligned}$$

The functions $\eta_k(s)$ and $e_k(s)$ have already been studied in [27], where it was shown that

$$\eta_k(s), e_k(s) \in L^2(dm(s))$$

and, with respect to this inner product,

$$\sum_{k=0}^{\infty} |e_k| \cdot |\eta_k| < \infty.$$

Then we have

$$\begin{aligned} \mathcal{L}_{T,\mu} f(x, y) &= \int_0^{\infty} \left(\frac{1}{x^2} \right) e^{-t(\frac{1-x+y}{x})} \left(\sum_{k=0}^{\infty} \langle \phi(x, s), \eta_k(s) \rangle e_k(t) \right) dm(t) \\ &= \sum_{k=0}^{\infty} \langle \phi(x, s), \eta_k(s) \rangle \int_0^{\infty} \left(\frac{1}{x^2} \right) e^{-t(\frac{1-x+y}{x})} e_k(t) dm(t) \\ &= \sum_{k=0}^{\infty} \langle \phi(x, s), \eta_k(s) \rangle E_k(x, y), \end{aligned}$$

where we set

$$E_k(x, y) = \int_0^{\infty} \left(\frac{1}{x^2} \right) e^{-t(\frac{1-x+y}{x})} e_k(t) dm(t).$$

We want to show that $E_k(x, y) \in L^2(dy)$, again treating the variable x as a parameter. (The calculation below will also yield that $E_k(x, y) \notin L^2(dx dy)$.)

We have

$$\begin{aligned} E_k(x, y) &= \int_0^{\infty} \left(\frac{1}{x^2} \right) e^{-t(\frac{1-x+y}{x})} L_n^1(t) dm_2(t) \\ &= \int_0^{\infty} \left(\frac{1}{x^2} \right) e^{-t(\frac{1-x+y}{x})} L_n^1(t) \frac{t}{e^t - 1} dt \\ &= \left(\frac{1}{x^2} \right) \int_0^{\infty} e^{-t(\frac{1-x+y}{x})} L_n^1(t) \frac{te^{-t}}{1 - e^{-t}} dt \\ &= \left(\frac{1}{x^2} \right) \int_0^{\infty} e^{-t(\frac{1-x+y}{x})} L_n^1(t) te^{-t} \sum_{m=0}^{\infty} e^{-mt} \\ &= \left(\frac{1}{x^2} \right) \int_0^{\infty} \sum_{m=0}^{\infty} t L_n^1(t) e^{-s(m)t} \\ &\quad \text{where } s(m) = \frac{1+y+mx}{x} \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{1}{x^2}\right) \sum_{m=0}^{\infty} \int_0^{\infty} t L_n^1(t) e^{-s(m)t} \\
&= \left(\frac{1}{x^2}\right) \sum_{m=0}^{\infty} \frac{(n+1)(s(m)-1)^n}{s(m)^{n+2}} \\
&\quad \text{by 7.414.8 in [14]} \\
&= (n+1) \left(\frac{1}{x^2}\right) \sum_{m=0}^{\infty} \frac{\left(\frac{1+y+mx}{x} - 1\right)^n}{\left(\frac{1+y+mx}{x}\right)^{n+2}} \\
&= (n+1) \sum_{m=0}^{\infty} \frac{(1+y+(m-1)x)^n}{(1+y+mx)^{n+2}}
\end{aligned}$$

As the above series grows like $\sum \frac{1}{m^2 x^2}$, there is a constant C independent of n such that

$$(n+1) \sum_{m=0}^{\infty} \frac{(1+y+(m-1)x)^n}{(1+y+mx)^{n+2}} < (n+1)C \left(\frac{1+y}{x^2}\right).$$

6 Conclusion

Of course traditional continued fractions have many rich properties. It is natural to ask which of these properties have analogs for a given multidimensional continued fraction algorithm. This paper is the start of finding the transfer operator analogs for the triangle map. In [3], similar analogs will be developed for triangle partition maps, which as mentioned, while being built in a natural way out of the triangle map, include many if not most multidimensional continued fraction algorithms.

As mentioned in the introduction, transfer operator have been used for decades in the study of continued fractions, though people did not initially use the framework or rhetoric of functional analysis. The framework was made

explicit in the pioneering work of Mayer and Roepstroff [27, 28] and of Mayer [25, 26], work that continues even today [36, 37, 19, 4, 20, 12, 16, 8, 1, 17, 7, 6]. For each of these papers, there are corresponding natural questions for multidimensional continued fractions. The corresponding questions should be non-trivial but interesting.

References

- [1] G. Alkauskas, Transfer operator for the Gauss' continued fraction map. I. Structure of the eigenvalues and trace formulas, <http://arxiv.org/pdf/1210.4083v6.pdf>
- [2] I. Amburg, *Explicit forms for and some functional analysis behind a family of multidimensional continued fractions : triangle partition maps and their associated transfer operators*, Thesis, Williams College, 2014.
- [3] I. Amburg and T. Garrity, On Gauss-Kuzmin Statistics and the Transfer Operator for a Multidimensional Continued Fraction Algorithm: Triangle Partition Maps, in preparation.
- [4] I. Antoniou and S. A. Shkarin, Analyticity of smooth eigenfunctions and spectral analysis of the Gauss map, *Journal of Statistical Physics*, 111, no. 1-2, (2003), pp. 355-369.
- [5] V. Baladi, *Positive Transfer Operators and Decay of Correlation (Advanced Series in Nonlinear Dynamics)*, World Scientific, 2000.

- [6] *S. Ben Ammou, C. Bonanno, I. Chouari, S. Isola, On the spectrum of the transfer operators of a one-parameter family with intermittency transition, <http://lanl.arxiv.org/pdf/1506.02573v1.pdf>*
- [7] *,C. Bonanno and S. Isola, A thermodynamic approach to two-variable Ruelle and Selberg zeta functions via the Farey map, Nonlinearity 27 (2014), no. 5, pp.897-926.*
- [8] *C. Bonanno, S. Graffi, and S. Isola, Spectral analysis of transfer operators associated to Farey fractions, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 19 (2008), no. 1, pp. 1-23.*
- [9] *S. Assaf, L. Chen, T. Cheslack-Postava, B. Cooper, A. Diesl, T. Garrity, M. Lepinski and A. Schuyler, Dual Approach to Triangle Sequences: A Multidimensional Continued Fraction Algorithm, Integers, Vol. 5, (2005).*
- [10] *K. Dasaratha, L. Flapan, T. Garrity, C. Lee, C. Mihaila, N. Neumann-Chun, S. Peluse, and M. Stoffregen, A generalized family of multidimensional continued fractions: triangle partition maps, International Journal of Number Theory 10 (2014), pp. 2151–2186.*
- [11] *K. Dasaratha, L. Flapan, T. Garrity, C. Lee, C. Mihaila, N. Neumann-Chun, S. Peluse, and M. Stoffregen, Cubic irrationals and periodicity via a family of multi-dimensional continued fraction algorithms, Monatshefte für Mathematik 174 (2014), pp. 549–566.*

- [12] M. Degli Esposti, S. Isola and A. Knauf, *Generalized Farey trees, transfer operators and phase transitions*, *Communications in Mathematical Physics*, 275 (2007), no. 2, pp. 297-329.
- [13] T. Garrity, *On periodic sequences for algebraic numbers*, *J. of Number Theory*, 88, no. 1 (2001), pp. 83-103.
- [14] I. Gradshteyn and I. Ryshik, *Table of Integrals, Series, and Products*, *Corrected and Enlarged Edition prepared by A. Jeffrey, incorporating the fourth edition prepared by Y. Geraniums and M Tseytlin*, Academic Press, 1980.
- [15] D. Hensley, *Continued Fractions*, World Scientific, 2006.
- [16] J. Hilgert, *Mayer's transfer operator and representations of GL_2* , *Semigroup Forum* 77 (2008), no. 1, pp. 64-85.
- [17] M. Iosifescu, *Spectral analysis for the Gauss problem on continued fractions*, *Indag. Math (N.S.)* 25 (2014), no. 4, pp. 825-831.
- [18] M. Iosifescu and C. Kraaikamp, *Metrical Theory of Continued Fractions*, Kluwer Academic, 2002.
- [19] S. Isola, *On the spectrum of Farey and Gauss maps*. *Nonlinearity* 15, no. 5 (2002), pp. 1521-1539.
- [20] O. Jenkinson, L. Gonzalez and M. Urbanski, *On transfer operators for continued fractions with restricted digits*, *Proceeding of the London*

Mathematical Society, Proc. London Math. Soc. (3) 86 , no. 3, (2003), pp. 755-778.

[21] *O. Karpenkov, Geometry of Continued Fractions, Algorithms and Computations in Mathematics, vol. 26, (2013), Springer-Verlag.*

[22] *A. Khinchin, Continued Fractioins, Dover, 1997.*

[23] *M. A. Krsnodsel'skiĭ, Positive Solutions of Operator Equations, (translated by Richard E. Flaherty), P. Noordhoff LTD.-Groningen, the Netherlands.*

[24] *A. Lasota and M. Mackey, Probabilistic Properties of Deterministic Systems, Cambridge University Press, 1985.*

[25] *D. Mayer, On the thermodynamic formalism for the Gauss map, Communications in Mathematical Physics, 130 (1990), no. 2, pp. 311-333.*

[26] *D. Mayer, Continued fractions and related transformations. Ergodic theory, symbolic dynamics, and hyperbolic spaces (Trieste, 1989), Oxford Sci. Publ., Oxford Univ. Press, New York, 1991, pp. 175-222.*

[27] *D. Mayer and G. Roepstorff, On the Relaxation Time of Gauss's Continued-Fraction Map I: The Hilbert Space Approach (Koopmanism), Journal of Statistical Physics, Vol. 47, nos. 1/2 (1987), pp. 149-170.*

[28] *D. Mayer and G. Roepstorff, On the Relaxation Time of Gauss's Continued-Fraction Map II. The Banach space approach (transfer oper-*

- ator method), *Journal of Statistical Physics*, Vol. 50, no. 1-2, (1988), pp. 331-344.
- [29] A. Messaoudi, A. Nogueira and F. Schweiger, *Ergodic properties of triangle partitions*, *Monatsh. Math.* 157 (2009), no. 3, pp. 283-299.
- [30] A. Rockett and P. Szusz, *Continued Fractions*, World Scientific, 1992.
- [31] F. Schweiger, *Periodic multiplicative algorithms of Selmer type*, *Integers* 5 (2005), no. 1, A28.
- [32] F. Schweiger, *The metrical theory of Jacobi-Perron algorithm*, *Lecture Notes in Mathematics*, 334, Springer-Verlag, Berlin, 1973.
- [33] F. Schweiger, *Über einen Algorithmus von R. Güting*, *J. Reine Angew. Math.* 293/294 (1977), pp. 263-270.
- [34] F. Schweiger, *Ergodic Theory of Fibred Systems and Metric Number Theory*, Oxford University Press, Oxford, 1995.
- [35] F. Schweiger, *Multidimensional Continued Fractions*, Oxford University Press, 2000.
- [36] B. Vallée, *Opérateurs de Ruelle-Mayer généralisés et analyse en moyenne des algorithmes d'Euclide et de Gauss*, *Acta Arithmetica*, 81, no. 2, 1997, pp 101-144.
- [37] B. Vallée, *Dynamique des fractions continues à contraintes périodiques*, *Journal of Number Theory* 72, no. 2, (1998), pp.183-235